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Improved Closed-Form Approximation for Dutch Roll

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An improved Dutch-roll approximation that includes the effects of roll as well as those of sideslip and yaw is presented. This new approximation is based on a Taylor series expansion in one over the roll-damping derivative. The eigenvalues obtained from this solution are identical to those obtained from the traditional Dutch-roll approximation when the roll-damping derivative approaches infinity. From the new closed-form approximation, the Dutch-roll frequency is shown to be a function of a dimensionless parameter, which the author has called the Dutch-roll stability ratio. In addition, this new solution shows that there are three distinct components to the Dutch-roll damping. The first is the conventional yaw damping term, but the present solution points out two other contributions to the Dutch-roll damping. These are called the Dutch-roll coupling and phase damping. In most cases, the yaw damping is the largest of these three components. However, both the coupling and the phase damping can degrade the total Dutch-roll damping and, under certain conditions, could cause the Dutch-roll motion to become divergent.

Nomenclature

A_w = planform area of the wing
b = wingspan
C_l = rolling moment coefficient
$C_{l,\bar{p}}$ = change in rolling moment coefficient with dimensionless rolling rate
$C_{l,\bar{r}}$ = change in rolling moment coefficient with dimensionless yawing rate
$C_{l,\beta}$ = change in rolling moment coefficient with sideslip angle
C_n = yawing moment coefficient
$C_{n,\bar{p}}$ = change in yawing moment coefficient with dimensionless rolling rate
$C_{n,\bar{r}}$ = change in yawing moment coefficient with dimensionless yawing rate
$C_{n,\beta}$ = change in yawing moment coefficient with sideslip angle
C_Y = side-force coefficient
$C_{Y,\bar{p}}$ = change in side-force coefficient with dimensionless rolling rate
$C_{Y,\bar{r}}$ = change in side-force coefficient with dimensionless yawing rate
$C_{Y,\beta}$ = change in side-force coefficient with sideslip angle
g = acceleration of gravity
I_{xx} = rolling moment of inertia, body-fixed coordinates
I_{zz} = yawing moment of inertia, body-fixed coordinates
m = aircraft mass
p = rolling rate
$\bar{p}$ = dimensionless rolling rate
R_{Dc} = Dutch-roll-coupling ratio
R_{Dp} = Dutch-roll phase-divergence ratio
R_{Ds} = Dutch-roll-stability ratio
R_{dc} = complex coefficient
R_{fc} = complex coefficient
R_{gy} = dimensionless gravitational ratio
$R_{l,\bar{p}}$ = dimensionless roll-damping ratio
$R_{l,\bar{r}}$ = dimensionless roll-coupling ratio
$R_{l,\beta}$ = dimensionless roll-stability ratio
$R_{n,\bar{p}}$ = dimensionless yaw-coupling ratio
$R_{n,\bar{r}}$ = dimensionless yaw-damping ratio
$R_{n,\beta}$ = dimensionless yaw-stability ratio
$R_{Y,\bar{p}}$ = dimensionless rolling side-force ratio
$R_{Y,\bar{r}}$ = dimensionless yawing side-force ratio
$R_{Y,\beta}$ = dimensionless slip-damping ratio
r = yawing rate
$\bar{r}$ = dimensionless yawing rate
t = time
V = airspeed
V_0 = equilibrium airspeed
v = sideslip velocity component, body-fixed coordinates
W = aircraft weight
y_f = spanwise position, Earth-fixed coordinates
β = sideslip angle
Δ = deviation from equilibrium
λ = eigenvalue
λ_∞ = eigenvalue for infinite roll-damping ratio
ξ = dimensionless spanwise position, Earth-fixed coordinates
ρ = air density
σ_{Dr} = Dutch-roll-damping rate
$\bar{\sigma}_\infty$ = dimensionless Dutch-roll-damping for infinite roll damping
τ = dimensionless time
ϕ = Euler bank angle or roll attitude
ψ = Euler azimuth angle or heading
ω_{Dr} = Dutch-roll-damped natural frequency
$\bar{\omega}_\infty$ = dimensionless undamped Dutch-roll frequency for infinite roll damping

Introduction

DUTCH roll is a lightly damped oscillatory motion that can occur when an aircraft is disturbed laterally from equilibrium flight. The motion is an oscillatory interchange between sideslip, roll, and yaw that occurs as a statically stable aircraft attempts to reestablish lateral and directional equilibrium. Dutch roll is usually the most troublesome of the natural modes associated with the dynamics of an aircraft in free flight. This periodic motion has been studied for nearly 100 years and is well understood.

One approach to the study of aircraft dynamics involves solving the full nonlinear equations of motion.[1] This system of nonlinear equations is quite complex. Nonlinear dynamics with three degrees of freedom is typically treated using the method of bifurcation together with numerical methods. Campos[2] gives a good review of the work on nonlinear aircraft dynamics.

A more common approach to aircraft dynamics starts with the linearized equations of motion that were first developed by Bryan.[3] These linearized equations result in an eigenproblem of order twelve

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that can be solved for the dynamic modes associated with an aircraft in free flight. Perkins[4] has presented a detailed review of the early work on linearized aircraft dynamics.

Linearized Dutch roll is characterized by the frequency, the damping rate, and the relative amplitudes and phase shifts for the oscillations in sideslip, rolling rate, yawing rate, course deviation, bank angle, and heading. Once the aerodynamic stability and damping derivatives have been determined from wind-tunnel tests or other means, the free-flight Dutch-roll characteristics for an aircraft can readily be evaluated. This can be done by numerically determining the eigenvalues and eigenvectors associated with the linearized equations of motion (see, for example, Etkin and Reid,[5] McCormick,[6] McRuer et al.,[7] Nelson,[8] or Perkins and Hage[9]). However, the eigenvalues and eigenvectors for Dutch roll depend on many aircraft design and operating parameters, and the nature of this dependence is not easily observable from a numerical solution. For this reason, a closed-form approximation that accurately describes the essential features of Dutch roll is desirable. In addition, closed-form solutions have always been useful for the optimization of aircraft control systems (see Ashkenas and McRuer[10]).

The usual closed-form approximation for Dutch roll is obtained by assuming the motion consists solely of sideslip and yaw. The rolling rate, the bank angle, and the rolling momentum equation are all completely neglected in the traditional Dutch-roll approximation. Results obtained from this commonly used approximation are not in particularly good agreement with the exact solution for Dutch roll. The reason for poor agreement between this traditional approximation and the exact solution is that Dutch roll involves significant oscillations in bank angle. Although several variations of the traditional Dutch-roll approximation have been proposed, none of these accurately predict many of the fundamental characteristics of Dutch roll. Both McRuer et al.[7] and, more recently, Etkin and Reid[5] present good reviews of existing Dutch-roll approximations. In the present paper, an improved Dutch-roll approximation that includes the effects of roll as well as those of sideslip and yaw is presented. This new Dutch-roll approximation is based on the method used by Phillips[11] for the analysis of phugoid motion.

Traditional Dutch-Roll Approximation

To obtain the Dutch-roll eigenvalues and eigenvectors, whether numerically or analytically, we start with the linearized lateral equations of aircraft motion. The development of these equations can be found in any undergraduate textbook dealing with aircraft dynamics.[5–9] The eigenvalues and eigenvectors are obtained from the homogeneous equations with all control inputs set to zero. For the Dutch-roll approximation, we neglect the product of inertia and the change in side force with respect to rolling rate because these values are typically small and have little effect on Dutch roll. If we also restrict the analysis to deviations about level flight, the familiar linearized lateral equations of motion can be written in dimensionless form as

$$\begin{Bmatrix} \Delta\hat{\beta} \\ \Delta\hat{\bar{p}} \\ \Delta\hat{\bar{r}} \\ \Delta\hat{\xi} \\ \Delta\hat{\phi} \\ \Delta\hat{\psi} \end{Bmatrix} = \begin{bmatrix} R_{Y,\beta} & 0 & (R_{Y,\bar{r}}-1) & 0 & R_{gy} & 0 \\ R_{l,\beta} & R_{l,\bar{p}} & R_{l,\bar{r}} & 0 & 0 & 0 \\ R_{n,\beta} & R_{n,\bar{p}} & R_{n,\bar{r}} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\beta \\ \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} \tag{1}$$

where a subscript following a comma indicates differentiation. As is the usual convention, the characteristic length is taken to be the semispan and the characteristic velocity is taken to be the equilibrium airspeed. Thus,

$$\Delta\beta \cong \Delta v/V_0, \qquad \Delta\bar{p} \equiv b\Delta p/2V_0$$
$$\Delta\bar{r} \equiv b\Delta r/2V_0, \qquad \Delta\xi \equiv 2\Delta y_f/b \tag{2}$$

Here, the Δ indicates a deviation from equilibrium, and the notation used on the left-hand side of Eq. (1) indicates differentiation with respect to dimensionless time,

$$\hat{f} \equiv \frac{b}{2V_0}\frac{\partial f}{\partial t} \equiv \frac{\partial f}{\partial \tau}, \qquad \tau \equiv \frac{2V_0 t}{b} \tag{3}$$

The dimensionless coefficients on the right-hand side of Eq. (1) are all evaluated at the equilibrium flight condition and are defined as

$$R_{Y,\beta} \equiv (\rho A_w b/4m)C_{Y,\beta}, \qquad R_{Y,\bar{r}} \equiv (\rho A_w b/4m)C_{Y,\bar{r}}$$
$$R_{gy} \equiv gb/2V_0^2, \qquad R_{l,\beta} \equiv (\rho A_w b^3/8I_{xx})C_{l,\beta}$$
$$R_{l,\bar{p}} \equiv (\rho A_w b^3/8I_{xx})C_{l,\bar{p}}, \qquad R_{l,\bar{r}} \equiv (\rho A_w b^3/8I_{xx})C_{l,\bar{r}}$$
$$R_{n,\beta} \equiv (\rho A_w b^3/8I_{zz})C_{n,\beta}, \qquad R_{n,\bar{p}} \equiv (\rho A_w b^3/8I_{zz})C_{n,\bar{p}}$$
$$R_{n,\bar{r}} \equiv (\rho A_w b^3/8I_{zz})C_{n,\bar{r}} \tag{4}$$

Dutch roll is a damped oscillatory motion characterized by a combination of sideslip, roll, and yaw. Nevertheless, the best known and most widely used Dutch-roll approximation is a flat yawing/sideslipping motion in which roll is completely suppressed. In this approximation, the rolling rate, the bank angle, and the rolling momentum equation are completely neglected. Using these approximations for motion relative to level flight, the eigenproblem associated with Eq. (1) becomes

$$\begin{bmatrix} (R_{Y,\beta}-\lambda) & 0 & (R_{Y,\bar{r}}-1) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ R_{n,\beta} & 0 & (R_{n,\bar{r}}-\lambda) & 0 & 0 & 0 \\ 1 & 0 & 0 & -\lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\lambda \end{bmatrix} \begin{Bmatrix} \Delta\beta \\ \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5}$$

The nontrivial eigenvalues obtained from Eq. (5) are readily found to be

$$\lambda = (R_{Y,\beta} + R_{n,\bar{r}})/2 \pm i\sqrt{(1 - R_{Y,\bar{r}})R_{n,\beta} + R_{Y,\beta}R_{n,\bar{r}} - [(R_{Y,\beta} + R_{n,\bar{r}})/2]^2} \tag{6}$$

and the associated eigenvectors are given by

$$\begin{Bmatrix} \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_{n,\beta}/(\lambda - R_{n,\bar{r}}) \\ 1/\lambda + [R_{n,\beta}/(\lambda - R_{n,\bar{r}})]/\lambda^2 \\ 0 \\ [R_{n,\beta}/(\lambda - R_{n,\bar{r}})]/\lambda \end{Bmatrix} \Delta\beta \tag{7}$$

From Eq. (6), the Dutch-roll frequency is

$$\omega_{\mathrm{Dr}} = (2V_0/b)\,\mathrm{Im}(\lambda) = (2V_0/b)\sqrt{(1 - R_{Y,\bar{r}})R_{n,\beta} + R_{Y,\beta}R_{n,\bar{r}} - [(R_{Y,\beta} + R_{n,\bar{r}})/2]^2} \tag{8}$$

and the Dutch-roll damping rate is

$$\sigma_{\mathrm{Dr}} = -(2V_0/b)\,\mathrm{Re}(\lambda) = -(V_0/b)(R_{Y,\beta} + R_{n,\bar{r}}) \tag{9}$$

Best results from this traditional approximation are obtained for light aircraft. For example, consider the typical general aviation airplane having

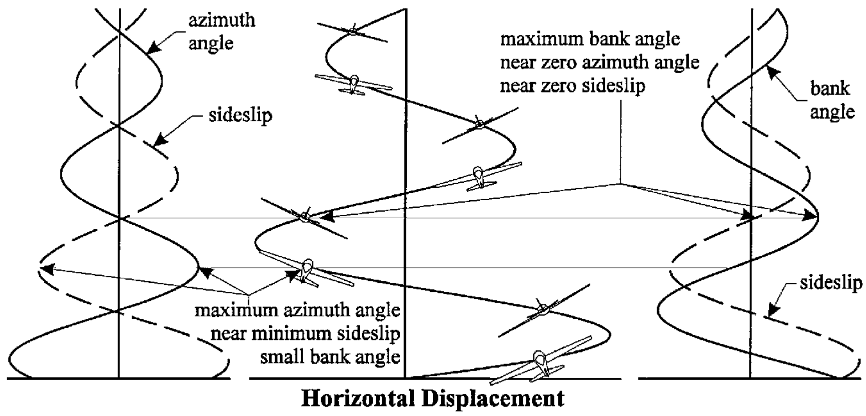

Fig. 1 Typical Dutch-roll flight deviations for a light airplane.

$$A_w = 185\ \text{ft}^2, \quad b = 33\ \text{ft}, \quad V_0 = 180\ \text{ft/s}$$

$$W = 2800\ \text{lbf}, \quad I_{xx} = 1000\ \text{slug-ft}^2, \quad I_{zz} = 3500\ \text{slug-ft}^2$$

$$C_{Y,\beta} = -0.560, \quad C_{Y,\bar{p}} = 0.0, \quad C_{Y,\bar{r}} = 0.240$$

$$C_{l,\beta} = -0.075, \quad C_{l,\bar{p}} = -0.410, \quad C_{l,\bar{r}} = 0.105$$

$$C_{n,\beta} = 0.070, \quad C_{n,\bar{p}} = -0.0575, \quad C_{n,\bar{r}} = -0.125$$

For this light airplane, the exact solution for the Dutch-roll eigenvalues and eigenvectors obtained numerically from Eq. (1) gives

$$\begin{Bmatrix} \Delta\beta \\ \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = C_{1,2} \begin{Bmatrix} 0.58853 \\ -0.11045 \mp 0.00147i \\ 0.01444 \mp 0.12146i \\ 0.16888 \mp 0.22197i \\ 0.09389 \pm 0.48749i \\ -0.54776 \pm 0.04678i \end{Bmatrix}$$

$$\times \exp[(-0.04498 \pm 0.21791i)\tau]$$

whereas the traditional approximation obtained from Eqs. (6) and (7) yields

$$\begin{Bmatrix} \Delta\beta \\ \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = C_{1,2} \begin{Bmatrix} 0.65348 \\ 0.00000 \\ 0.01558 \mp 0.12960i \\ 0.34429 \mp 0.13878i \\ 0.00000 \\ -0.64238 \pm 0.07410i \end{Bmatrix}$$

$$\times \exp[(-0.04690 \pm 0.19634i)\tau]$$

Even for this light aircraft, although the eigenvalues predicted by the traditional approximation are accurate to within about 10%, the eigenvectors do not confirm the underlying hypothesis that oscillations in roll are negligible. For light aircraft, the largest Dutch-roll oscillations are typically in sideslip. However, the Dutch-roll amplitude for the rolling rate is nearly as large as that for sideslip and is about the same as that for the yawing rate. Furthermore, Dutch-roll oscillations in bank angle are typically as large as those in heading. Figure 1 shows the relative amplitudes and phase shifts for the Dutch-roll flight deviations of a typical light airplane. Because Dutch roll in larger aircraft typically produces even larger rolling oscillations, it is difficult to justify neglecting roll when attempting to develop a Dutch-roll approximation.

Improved Dutch-Roll Approximation

We shall now develop an improved Dutch-roll approximation that includes the effects of roll as well as those of sideslip and yaw. Accounting for all three lateral degrees of freedom, the eigenproblem associated with Eq. (1) is

$$\begin{bmatrix} (R_{Y,\beta} - \lambda) & 0 & (R_{Y,\bar{r}} - 1) & 0 & R_{gy} & 0 \\ R_{l,\beta} & (R_{l,\bar{p}} - \lambda) & R_{l,\bar{r}} & 0 & 0 & 0 \\ R_{n,\beta} & R_{n,\bar{p}} & (R_{n,\bar{r}} - \lambda) & 0 & 0 & 0 \\ 1 & 0 & 0 & -\lambda & 0 & 1 \\ 0 & 1 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 1 & 0 & 0 & -\lambda \end{bmatrix} \times \begin{Bmatrix} \Delta\beta \\ \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{10}$$

Using the second and fifth equations in Eq. (10), we can eliminate the rolling rate and bank angle from the first and third equations. Thus, Eq. (10) can be rearranged and separated to yield

$$\begin{bmatrix} \left(R_{Y,\beta} - \dfrac{R_{gy} R_{l,\beta}}{(R_{l,\bar{p}} - \lambda)\lambda} - \lambda\right) & \left(-1 + R_{Y,\bar{r}} - \dfrac{R_{gy} R_{l,\bar{r}}}{(R_{l,\bar{p}} - \lambda)\lambda}\right) \\ \left(R_{n,\beta} - \dfrac{R_{l,\beta} - R_{n,\bar{p}}}{R_{l,\bar{p}} - \lambda}\right) & \left(R_{n,\bar{r}} - \dfrac{R_{l,\bar{r}} R_{n,\bar{p}}}{R_{l,\bar{p}} - \lambda} - \lambda\right) \end{bmatrix} \times \begin{Bmatrix} \Delta\beta \\ \Delta\bar{r} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{11}$$

and

$$\begin{Bmatrix} \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = \begin{Bmatrix} R_{xc} \\ R_{zc} \\ (1 + R_{zc}/\lambda)/\lambda \\ R_{xc}/\lambda \\ R_{zc}/\lambda \end{Bmatrix} \Delta\beta \tag{12}$$

where

$$R_{xc} \equiv \frac{R_{n,\beta} R_{l,\bar{r}} - R_{l,\beta}(R_{n,\bar{r}} - \lambda)}{(R_{l,\bar{p}} - \lambda)(R_{n,\bar{r}} - \lambda) - R_{l,\bar{r}} R_{n,\bar{p}}}$$

$$R_{zc} \equiv \frac{R_{l,\beta} R_{n,\bar{p}} - R_{n,\beta}(R_{l,\bar{p}} - \lambda)}{(R_{l,\bar{p}} - \lambda)(R_{n,\bar{r}} - \lambda) - R_{l,\bar{r}} R_{n,\bar{p}}} \tag{13}$$

The characteristic equation for Eq. (11) is

$$\begin{vmatrix} \left(R_{Y,\beta} - \dfrac{R_{gy} R_{l,\beta}}{(R_{l,\bar{p}} - \lambda)\lambda} - \lambda\right) & \left(-1 + R_{Y,\bar{r}} - \dfrac{R_{gy} R_{l,\bar{r}}}{(R_{l,\bar{p}} - \lambda)\lambda}\right) \\ \left(R_{n,\beta} - \dfrac{R_{l,\beta} R_{n,\bar{p}}}{R_{l,\bar{p}} - \lambda}\right) & \left(R_{n,\bar{r}} - \dfrac{R_{l,\bar{r}} R_{n,\bar{p}}}{R_{l,\bar{p}} - \lambda} - \lambda\right) \end{vmatrix} = 0 \tag{14}$$

which can be expanded to give

$$\lambda^2 - \left(R_{Y,\beta} + R_{n,\bar{r}} - \frac{R_{l,\bar{r}} R_{n,\bar{p}}}{R_{l,\bar{p}} - \lambda}\right)\lambda + (1 - R_{Y,\bar{r}}) R_{n,\beta} + R_{Y,\beta} R_{n,\bar{r}}$$
$$+ \frac{R_{l,\beta}[R_{gy} - (1 - R_{Y,\bar{r}}) R_{n,\bar{p}}] - R_{Y,\beta} R_{l,\bar{r}} R_{n,\bar{p}}}{R_{l,\bar{p}} - \lambda}$$
$$+ \frac{R_{gy}(R_{l,\bar{r}} R_{n,\beta} - R_{l,\beta} R_{n,\bar{r}})}{(R_{l,\bar{p}} - \lambda)\lambda} = 0 \tag{15}$$

Because roll is typically heavily damped, those terms in Eq. (15) that are inversely proportional to the roll-damping ratio are small, but not necessarily totally negligible. This suggests expanding Eq. (15) in terms of a Taylor series in one over the roll-damping ratio. For this purpose we use the expansion,

$$\frac{1}{R_{l,\bar{p}} - \lambda} = \frac{1}{R_{l,\bar{p}}} + \frac{\lambda}{R_{l,\bar{p}}^2} + \frac{\lambda^2}{R_{l,\bar{p}}^3} + \frac{\lambda^3}{R_{l,\bar{p}}^4} + \cdots \tag{16}$$

If we let λ_∞ represent the eigenvalues corresponding to an infinite roll-damping ratio $R_{l,\bar{p}} \to \infty$, then Eq. (15) gives

$$\lambda_\infty^2 - (R_{Y,\beta} + R_{n,\bar{r}})\lambda_\infty + (1 - R_{Y,\bar{r}}) R_{n,\beta} + R_{Y,\beta} R_{n,\bar{r}} = 0 \tag{17}$$

and the eigenvalues are the roots of this quadratic equation,

$$\lambda_\infty = (R_{Y,\beta} + R_{n,\bar{r}})/2$$
$$\pm i\sqrt{(1 - R_{Y,\bar{r}}) R_{n,\beta} + R_{Y,\beta} R_{n,\bar{r}} - [(R_{Y,\beta} + R_{n,\bar{r}})/2]^2} \tag{18}$$

Note that the eigenvalues obtained from Eq. (18) are identical to those obtained from Eq. (6). Thus, we see that the traditional Dutch-roll approximation is valid only in the limit as the roll-damping ratio approaches infinity. If the roll-damping ratio were large enough, the rolling motion associated with Dutch roll would be completely suppressed and the motion would indeed consist solely of sideslip and yaw. However, an aircraft seldom possesses sufficient roll damping to eliminate the rolling oscillations from Dutch roll. The effects of roll can be included in the Dutch-roll approximation by carrying some of the low order terms from the Taylor series expansion given by Eq. (16).

Because the roll-damping ratio is typically larger than the Dutch roll eigenvalue, the terms in Eq. (16) become smaller with increasing order. This Taylor series can be evaluated approximately by replacing the actual eigenvalues on the right-hand side with the approximate eigenvalues from Eq. (18),

$$\frac{1}{R_{l,\bar{p}} - \lambda} \cong \frac{1}{R_{l,\bar{p}}} + \frac{\lambda_\infty}{R_{l,\bar{p}}^2} + \frac{\lambda_\infty^2}{R_{l,\bar{p}}^3} + \frac{\lambda_\infty^3}{R_{l,\bar{p}}^4} + \cdots \tag{19}$$

Furthermore, because the Dutch-roll damping is typically an order of magnitude less than the Dutch-roll frequency, we can also neglect the damping in Eq. (19) and use the approximations,

$$\frac{1}{R_{l,\bar{p}} - \lambda} \cong \frac{1}{R_{l,\bar{p}}} \pm i\frac{\bar{\omega}_\infty}{R_{l,\bar{p}}^2} - \frac{\bar{\omega}_\infty^2}{R_{l,\bar{p}}^3} \mp i\frac{\bar{\omega}_\infty^3}{R_{l,\bar{p}}^4} + \cdots \tag{20}$$

and

$$\frac{1}{(R_{l,\bar{p}} - \lambda)\lambda} \cong \mp i\frac{1}{\bar{\omega}_\infty R_{l,\bar{p}}} + \frac{1}{R_{l,\bar{p}}^2} \pm i\frac{\bar{\omega}_\infty}{R_{l,\bar{p}}^3} - \frac{\bar{\omega}_\infty^2}{R_{l,\bar{p}}^4} + \cdots \tag{21}$$

where

$$\bar{\omega}_\infty \equiv \sqrt{(1 - R_{Y,\bar{r}}) R_{n,\beta} + R_{Y,\beta} R_{n,\bar{r}}} \tag{22}$$

Retaining the first two terms in each of these Taylor series and applying the results to the characteristic equation given by Eq. (15), we obtain a quadratic equation for the approximate Dutch-roll eigenvalues,

$$\lambda^2 - (R_{Y,\beta} + R_{n,\bar{r}} + R_{dc})\lambda + (1 - R_{Y,\bar{r}}) R_{n,\beta}$$
$$+ R_{Y,\beta} R_{n,\bar{r}} + R_{fc} = 0 \tag{23}$$

where the complex coefficients R_{dc} and R_{fc} are defined as

$$R_{dc} \equiv -\frac{R_{l,\bar{r}} R_{n,\bar{p}}}{R_{l,\bar{p}}} \mp i\frac{\bar{\omega}_\infty R_{l,\bar{r}} R_{n,\bar{p}}}{R_{l,\bar{p}}^2} \tag{24}$$

and

$$R_{fc} \equiv \left[\frac{R_{l,\beta}[R_{gy} - (1 - R_{Y,\bar{r}}) R_{n,\bar{p}}] - R_{Y,\beta} R_{l,\bar{r}} R_{n,\bar{p}}}{R_{l,\bar{p}}} + \frac{R_{gy}(R_{l,\bar{r}} R_{n,\beta} - R_{l,\beta} R_{n,\bar{r}})}{R_{l,\bar{p}}^2}\right]$$
$$\pm i\left[\frac{R_{gy}(R_{l,\beta} R_{n,\bar{r}} - R_{l,\bar{r}} R_{n,\beta})}{\bar{\omega}_\infty R_{l,\bar{p}}} + \frac{\bar{\omega}_\infty \{R_{l,\beta}[R_{gy} - (1 - R_{Y,\bar{r}}) R_{n,\bar{p}}] - R_{Y,\beta} R_{l,\bar{r}} R_{n,\bar{p}}\}}{R_{l,\bar{p}}^2}\right] \tag{25}$$

The eigenvalues obtained from Eq. (23), with the use of Eq. (22), can be written as

$$\lambda = (R_{Y,\beta} + R_{n,\bar{r}} + R_{dc})/2$$
$$\pm i\sqrt{(1 - R_{Y,\bar{r}}) R_{n,\beta} + R_{Y,\beta} R_{n,\bar{r}} + R_{fc} - [(R_{Y,\beta} + R_{n,\bar{r}} + R_{dc})/2]^2}$$
$$= (R_{Y,\beta} + R_{n,\bar{r}} + R_{dc})/2$$
$$\pm i\bar{\omega}_\infty\sqrt{1 + R_{fc}\big|\ \bar{\omega}_\infty^2 - [(R_{Y,\beta} + R_{n,\bar{r}} + R_{dc})/2\bar{\omega}_\infty]^2} \tag{26}$$

Because both the complex constant R_{fc} and the damping term are small compared to $\bar{\omega}_\infty$, Eq. (26) can be further approximated as

$$\lambda \cong \frac{R_{Y,\beta} + R_{n,\bar{r}} + R_{dc}}{2}$$
$$\pm i\bar{\omega}_\infty\left(1 + \frac{\mathrm{Re}(R_{fc})}{2\bar{\omega}_\infty^2} \pm i\frac{\mathrm{Im}(R_{fc})}{2\bar{\omega}_\infty^2} - \frac{(R_{Y,\beta} + R_{n,\bar{r}} + R_{dc})^2}{8\bar{\omega}_\infty^2}\right)$$
$$\cong \frac{R_{Y,\beta} + R_{n,\bar{r}} + R_{dc} - \mathrm{Im}(R_{fc})/\bar{\omega}_\infty}{2}$$
$$\pm i\varpi_\infty\sqrt{1 + \frac{\mathrm{Re}(R_{fc})}{\bar{\omega}_\infty^2} - \left(\frac{R_{Y,\beta} + R_{n,\bar{r}} + R_{dc}}{2\bar{\omega}_\infty}\right)^2}$$
$$= \frac{R_{Y,\beta} + R_{n,\bar{r}} + R_{dc} - \mathrm{Im}(R_{fc})/\bar{\omega}_\infty}{2}$$
$$\pm i\sqrt{\bar{\omega}_\infty^2 + \mathrm{Re}(R_{fc}) - \left(\frac{R_{Y,\beta} + R_{n,\bar{r}} + R_{dc}}{2}\right)^2} \tag{27}$$

Because the imaginary component of the Dutch-roll eigenvalue is typically an order of magnitude larger than the real component, the second-order terms in R_{dc} and R_{fc} do not have a significant effect on the Dutch-roll frequency. However, the second-order terms are crucial to an accurate determination of the much smaller Dutch-roll-damping rate. For this reason, we shall retain all second-order terms in the real component of the eigenvalue but neglect second-order terms in the imaginary component. Thus, because the imaginary component of R_{dc} is second order, it can be neglected in the Dutch-roll approximation. We can also neglect the second-order term in the real part of R_{fc} because it affects only the frequency. On the other hand, we must retain the second term in the imaginary part of R_{fc} because this term is part of the damping. Thus, for this approximation, the Dutch-roll eigenvalues are

$$\lambda \cong (R_{Y,\beta} + R_{n,\bar{r}} - R_{\mathrm{Dc}} + R_{\mathrm{Dp}})/2$$
$$\pm i\sqrt{(1 - R_{Y,\bar{r}})R_{n,\beta} + R_{Y,\beta}R_{n,\bar{r}} + R_{\mathrm{Ds}} - [(R_{Y,\beta} + R_{n,\bar{r}})/2]^2} \tag{28}$$

where we define the Dutch-roll-stability ratio,

$$R_{\mathrm{Ds}} \equiv \{R_{l,\beta}[R_{gy} - (1 - R_{Y,\bar{r}})R_{n,\bar{p}}] - R_{Y,\beta}R_{l,\bar{r}}R_{n,\bar{p}}\}/R_{l,\bar{p}} \tag{29}$$

the Dutch-roll-coupling ratio,

$$R_{\mathrm{Dc}} \equiv R_{l,\bar{r}}R_{n,\bar{p}}/R_{l,\bar{p}} \tag{30}$$

and the Dutch-roll phase-divergence ratio,

$$R_{\mathrm{Dp}} \equiv \frac{R_{gy}(R_{l,\bar{r}}R_{n,\beta} - R_{l,\beta}R_{n,\bar{r}})}{R_{l,\bar{p}}(R_{n,\beta} + R_{Y,\beta}R_{n,\bar{r}})} - \frac{R_{\mathrm{Ds}}}{R_{l,\bar{p}}} \tag{31}$$

This gives the approximate Dutch-roll frequency,

$$\omega_{\mathrm{Dr}} \cong (2V_0/b)$$
$$\times\sqrt{(1 - R_{Y,\bar{r}})R_{n,\beta} + R_{Y,\beta}R_{n,\bar{r}} + R_{\mathrm{Ds}} - [(R_{Y,\beta} + R_{n,\bar{r}})/2]^2} \tag{32}$$

and the approximate Dutch-roll-damping rate,

$$\sigma_{\mathrm{Dr}} \cong -(V_0/b)(R_{Y,\beta} + R_{n,\bar{r}} - R_{\mathrm{Dc}} + R_{\mathrm{Dp}}) \tag{33}$$

Results

In contrast to the result predicted from Eq. (8), the Dutch-roll frequency obtained from Eq. (32) depends on the roll stability of the aircraft and the acceleration of gravity. The Dutch-roll damping from Eq. (33) contains two additional contributions that are not included in the approximation given by Eq. (9). Here the Dutch-roll damping is found to be a function of the roll-damping derivative, the roll–yaw coupling derivatives, and the lateral stability derivatives as well as the yaw and slip-damping derivatives. The eigenvalues and eigenvectors predicted from Eqs. (28) and (12) are greatly improved over those predicted from Eqs. (6) and (7). For example, consider the typical midsize jet transport[8] at 35,000 ft having

$$A_w = 2000\ \mathrm{ft}^2, \qquad b = 120\ \mathrm{ft}, \qquad V_0 = 778.5\ \mathrm{ft/s}$$
$$M = 0.8, \qquad W = 126{,}000\ \mathrm{lbf}$$
$$I_{xx} = 1.15 \times 10^5\ \mathrm{slug \cdot ft^2}, \qquad I_{zz} = 4.07 \times 10^6\ \mathrm{slug \cdot ft^2}$$
$$C_{Y,\beta} = -0.812, \qquad C_{Y,\bar{p}} = 0.0, \qquad C_{Y,\bar{r}} = 0.0$$
$$C_{l,\beta} = -0.177, \qquad C_{l,\bar{p}} = -0.312, \qquad C_{l,\bar{r}} = 0.153$$
$$C_{n,\beta} = 0.129, \qquad C_{n,\bar{p}} = -0.011, \qquad C_{n,\bar{r}} = -0.165$$

For this airplane and flight condition, the exact solution for the dimensionless Dutch-roll eigenvalues and eigenvectors obtained numerically from Eq. (1) gives

$$\begin{Bmatrix} \Delta\beta \\ \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = C_{1,2}\begin{Bmatrix} 0.024277 \mp 0.180838i \\ -0.008964 \pm 0.103525i \\ -0.017147 \mp 0.002782i \\ -0.239543 \mp 0.093425i \\ 0.930224 \\ -0.011572 \pm 0.155079i \end{Bmatrix}$$
$$\times \exp[(-0.00964 \pm 0.11129i)\tau]$$

The approximate solution obtained by using Eq. (28) with Eq. (12) results in

$$\begin{Bmatrix} \Delta\beta \\ \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = C_{1,2}\begin{Bmatrix} 0.02420 \mp 0.18095i \\ -0.00890 \pm 0.10357i \\ -0.01714 \mp 0.00278i \\ -0.24199 \mp 0.09172i \\ 0.92977 \\ -0.01167 \pm 0.15487i \end{Bmatrix}$$
$$\times \exp[(-0.00957 \pm 0.11140i)\tau]$$

whereas the approximate solution obtained from Eqs. (6) and (7) yields

$$\begin{Bmatrix} \Delta\beta \\ \Delta\bar{p} \\ \Delta\bar{r} \\ \Delta\xi \\ \Delta\phi \\ \Delta\psi \end{Bmatrix} = C_{1,2}\begin{Bmatrix} 0.59697 \\ 0.00000 \\ 0.00112 \mp 0.06001i \\ 0.52363 \mp 0.11664i \\ 0.00000 \\ -0.59104 \pm 0.05393i \end{Bmatrix}$$
$$\times \exp[(-0.01106 \pm 0.10053i)\tau]$$

For this jet transport, the eigenvalues and eigenvectors predicted by the improved Dutch-roll approximation agree very closely with the exact solution. For the traditional approximation the damped natural frequency is in error by 10%, the damping error is 15%, and the eigenvectors are not even reasonable. Notice that, for this midsize aircraft, the largest Dutch-roll oscillations are in bank angle, which is completely neglected in the traditional approximation.

Figures 2–7 show how the Dutch-roll approximation given by Eq. (28) compares with that given by Eq. (6) and the exact solution for a broad range of aerodynamic parameters. Note that, as the roll-damping derivative approaches infinity, the Dutch-roll-stability ratio, the Dutch-roll-coupling ratio, and the Dutch-roll

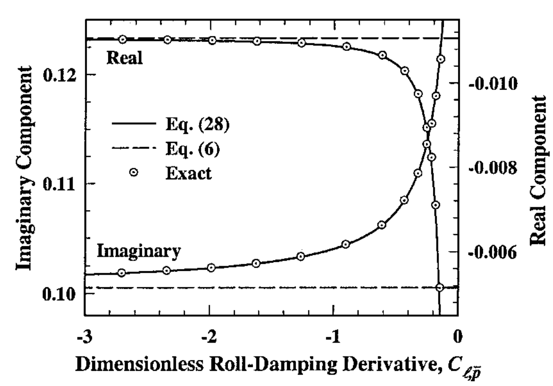

Fig. 2 Effect of roll damping on the dimensionless Dutch-roll eigenvalues.

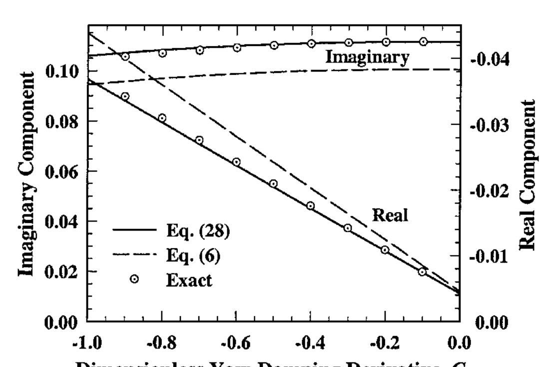

Fig. 3 Effect of yaw damping on the dimensionless Dutch-roll eigenvalues.

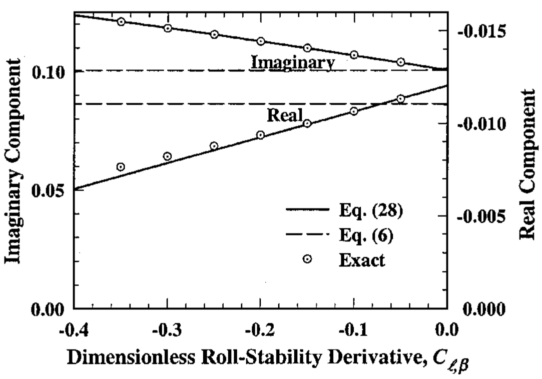

Fig. 4 Effect of roll stability on the dimensionless Dutch-roll eigenvalues.

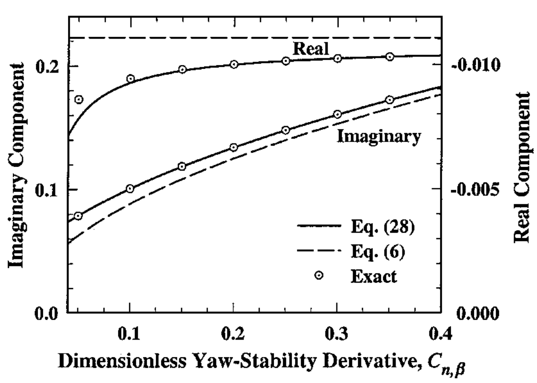

Fig. 5 Effect of yaw stability on the dimensionless Dutch-roll eigenvalues.

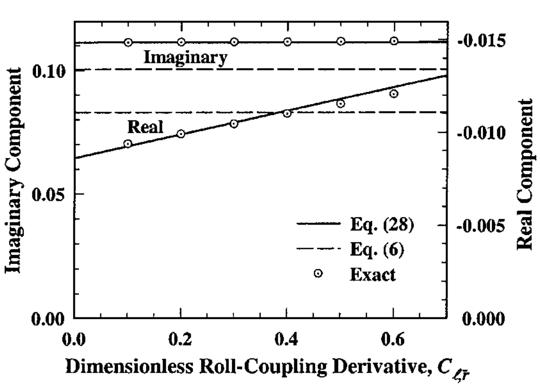

Fig. 6 Effect of roll coupling on the dimensionless Dutch-roll eigenvalues.

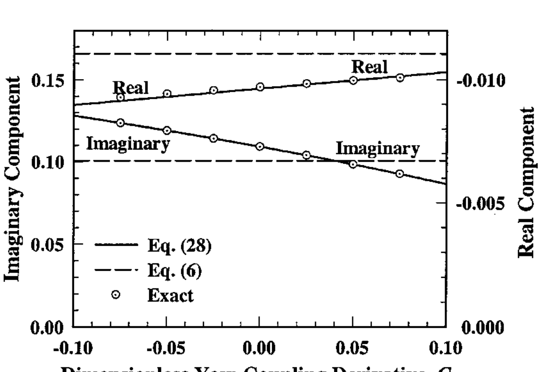

Fig. 7 Effect of yaw coupling on the dimensionless Dutch-roll eigenvalues.

phase-divergence ratio all approach zero. Hence, we see that, in the limit of infinite roll damping, the result given by Eq. (28) reduces exactly to that given by Eq. (6). Thus, as mentioned earlier, Eq. (6) represents an asymptotic solution for infinite roll damping. This can be seen graphically in Fig. 2, where all parameters except the roll-damping derivative have been held constant at those values given in the aforementioned example. In Fig. 3, a comparison between Eq. (6), Eq. (28), and the exact solution is shown for a broad range of yaw damping. In Fig. 4, the same comparison is shown for a broad range of roll-stability derivatives. Figure 5 shows the same comparison for a broad range of yaw-stability derivative. Similar comparisons, showing the effects of roll and yaw coupling are displayed in Figs. 6 and 7, respectively.

The aircraft chosen for the example given was a midsize jet transport. The author has made similar comparisons for other types of aircraft. Light aircraft have a low rolling moment of inertia, relatively high roll damping, and are typically flown at lower altitudes. Thus, light aircraft are characterized by a fairly high roll-damping ratio and are particularly flattering to the traditional Dutch-roll approximation. For such aircraft, the traditional approximation gives eigenvalues that are typically accurate to within about 10%, and the improved approximation is accurate to within a small fraction of 1%.

For larger aircraft, with higher rolling moment of inertia and higher cruise altitude, the roll-damping ratio is typically lower, and both the traditional Dutch-roll approximation and the improved approximation become worse. As the roll-damping ratio is reduced, the Dutch-roll eigenvalues begin to deviate more rapidly from those predicted by the traditional approximation. For example, if we double the rolling moment of inertia of the aircraft in the earlier example, the roll-damping ratio is cut in half. In this case, the exact dimensionless Dutch-roll eigenvalues are

$$\lambda_{\mathrm{Dr}} = -0.00839 \pm 0.11090i$$

The approximate solution obtained from Eq. (28) gives

$$\lambda_{\mathrm{Dr}} = -0.00824 \pm 0.11140i$$

whereas the approximate solution obtained from Eq. (6) results in

$$\lambda_{\mathrm{Dr}} = -0.01106 \pm 0.10053i$$

With this increase in rolling moment of inertia, the damping error for the traditional Dutch-roll approximation is increased to more than 30%, and the damping error for the improved Dutch-roll approximation is increased to almost 2%.

Because the traditional Dutch-roll approximation requires an infinite roll-damping ratio and the improved approximation is based on a large but finite roll-damping ratio, we should expect the accuracy of both approximations to deteriorate as the roll-damping ratio is decreased. Because the roll-damping ratio is inversely proportional to rolling moment of inertia and directly proportional to air density, a very large airplane flying at very high altitude can result in a low roll-damping ratio, which may invalidate both the traditional Dutch-roll approximation and the current improved approximation. The actual Dutch-roll eigenvalues deviate substantially from the traditional approximation whenever the roll-damping ratio is less than about 1.0. The improved Dutch-roll approximation that has been presented here is in good agreement with the exact solution for roll-damping ratios as low as about 0.3. However, for an aircraft having a roll-damping ratio less than about 0.3, the basis for the present approximation is no longer valid and this approximation should not be used. Nevertheless, a very wide variety of small to large airplanes have roll-damping ratios in the range from 0.3 to 1.0. For such aircraft the current Dutch-roll approximation provides a significant improvement over the traditional approximation.

Discussion

The present closed-form approximation allows us to see more easily how the aerodynamic coefficients and stability derivatives affect the Dutch-roll motion. Damping has very little effect on the

Dutch-roll frequency. Thus, neglecting the damping in Eq. (32) and applying the definition of Dutch-roll-stability ratio from Eq. (29), the undamped natural frequency for Dutch-roll motion is approximated as

$$\omega_{\text{Dr}} \cong (2V_0/b)\sqrt{\bar{\omega}_\infty^2 + R_{\text{Ds}}} = (2V_0/b) \times\sqrt{\bar{\omega}_\infty^2 + \{R_{l,\beta}[R_{gy} - (1 - R_{Y,\bar{r}})R_{n,\bar{p}}] - R_{Y,\beta}R_{l,\bar{r}}R_{n,\bar{p}}\}/R_{l,\bar{p}}} \tag{34}$$

where $\bar{\omega}_\infty$ is the traditional dimensionless undamped natural frequency for infinite roll damping. The roll-damping derivative is always negative and, for a stable aircraft, the roll-stability derivative is also negative. The gravitational acceleration ratio is positive, the dimensionless change in side force with yawing rate is always less than unity, and, for a conventional airplane generating positive lift, the change in yawing moment with rolling rate is negative. Thus, the Dutch-roll stability ratio is positive and, from Eq. (34), we see that increasing roll stability will increase the Dutch-roll frequency whereas increasing roll damping will decrease the Dutch-roll frequency. Furthermore, because the change in side force with sideslip angle is negative and the change in rolling moment with yawing rate is typically positive, Eq. (34) shows that increasing the roll–yaw coupling will normally increase the Dutch-roll frequency.

Using the definitions from Eqs. (29–31) in Eq. (33), the Dutch-roll damping for the present approximation can be written as

$$\sigma_{\text{Dr}} \cong \frac{V_0}{b}(\bar{\sigma}_\infty + R_{\text{Dc}} - R_{\text{Dp}}) = \frac{V_0}{b}\left(\bar{\sigma}_\infty + \frac{R_{l,\bar{r}}R_{n,\bar{p}}}{R_{l,\bar{p}}} - \frac{R_{gy}(R_{l,\bar{r}}R_{n,\beta} - R_{l,\beta}R_{n,\bar{r}})}{R_{l,\bar{p}}(R_{n,\beta} + R_{Y,\beta}R_{n,\bar{r}})} + \frac{R_{l,\beta}[R_{gy} - (1 - R_{Y,\bar{r}})R_{n,\bar{p}}] - R_{Y,\beta}R_{l,\bar{r}}R_{n,\bar{p}}}{R_{l,\bar{p}}^2}\right) \tag{35}$$

where $\bar{\sigma}_\infty$ is the traditional yaw-damping term that provides the total Dutch-roll damping for the case of infinite roll damping. The second term on the right-hand side of Eq. (35) is the Dutch-roll-coupling ratio. Because the roll–yaw coupling derivatives typically have opposite signs and the roll-damping derivative is always negative, the Dutch-roll-coupling ratio will normally increase the Dutch-roll damping. However, it is possible for the roll–yaw coupling derivatives to have the same sign, in which case this Dutch-roll coupling would tend to decrease the total Dutch-roll damping.

The third and fourth terms on the right-hand side of Eq. (35) compose the Dutch-roll phase-divergence ratio. This has been called the phase-divergence ratio because it results from the phase shift between the various components that comprise the Dutch-roll motion and because it will typically decrease the total Dutch-roll damping. For an aircraft that is stable in yaw, the denominator in the third term on the right-hand side of Eq. (35) is negative, and the denominator in the last term is always positive. Thus, because the roll-stability derivative is negative for a stable aircraft, Eq. (35) shows that increasing roll stability will decrease the Dutch-roll damping. In a similar manner, it can be shown that increasing yaw stability will increase the Dutch-roll damping. Because the phase divergence ratio is typically positive and decreases in magnitude with increasing roll damping, Eq. (35) also shows that increasing roll damping will normally increase the total Dutch-roll damping.

Conclusions

An improved closed-form approximation for Dutch roll has been developed. The results show that the traditional Dutch-roll approximation is an asymptotic solution that is valid only in the limit of infinite roll damping. As the roll damping approaches infinity, the eigenvalues and eigenvectors predicted by the present approximation approach those predicted by the traditional approximation. However, the level of roll damping that is required to converge the two approximations far exceeds the levels normally associated with typical aircraft. The new closed-form approximation points out two additional contributions to the Dutch-roll damping that the author has called Dutch-roll coupling and Dutch-roll phase damping. The Dutch-roll phase damping is a function of aircraft mass and moment of inertia, as well as the roll-damping and roll-stability derivatives. For a statically stable aircraft, the Dutch-roll phase damping is typically negative, tending to decrease the total Dutch-roll damping, and it is possible for this negative phase damping to render the Dutch-roll motion divergent. It has been shown that the traditional Dutch-roll approximation requires infinite roll damping and should not be used when the roll-damping ratio is less than about 1.0. The improved approximation is based on large but finite roll damping and should not be used when the roll-damping ratio is less than about 0.3.

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